Holomorphic dynamical systems.

References:
Beardon: Iteration of rational functions.
Milnor: Dynamics in one complex variable
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1) Introduction.

Setting: X complex space (manifold). For us $X = \mathbb{C}$ or
$X = \hat{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

$f : X \circlearrowleft$ holomorphic map.

Examples: 1) $f : \mathbb{C} \to \mathbb{C}$, $f(z) = P(z) = a_0 + a_1 z + \dots + a_d z^d$ $P \in \mathbb{C}[z]$.

2) $f : \hat{\mathbb{C}} \circlearrowleft$ $f(z) = \dfrac{P(z)}{Q(z)}$ (rational function) $P, Q \in \mathbb{C}[z]$

(P, Q without common factors, i.e. $\{P = 0\} \cap \{Q = 0\} = \emptyset$).

Recall: $\deg P = d$ (if $a_d \neq 0$); $\deg\left(\frac{P}{Q}\right) = \max\{\deg P, \deg Q\}$.

f is defined on $\mathbb{C} \setminus \{Q = 0\}$.

We set $f(z_0) = \infty$ if $Q(z_0) = 0$, and $f(\infty) = \lim_{z \to \infty} f(z)$

$\left(= \frac{a_d}{b_d}\right.$ if $\deg P = \deg Q = d$, 0 if $\deg P < \deg Q$, ∞ if $\left.\deg P > \deg Q\right)$

(*)
Goal: study the behavior of the iterates of f.

$$f^{\circ n} \ (= f^n) = \underbrace{f \circ \dots \circ f}_{n \text{ times}} \qquad f^n : X \circlearrowleft.$$

More precisely: given $z_0 \in X$, set $z_1 = f(z_0), \dots z_n = f^n(z_0)$.

How does the sequence $(z_n)_{n \in \mathbb{N}}$ depend from the starting point z_0?

Let us study an example.

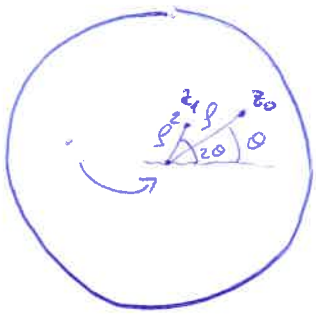
(*): Other examples: $f : \mathbb{C} \to \mathbb{C}$, $f(z) = e^z$ or $\cos z$

Exemple: $f : \mathbb{C} \to \mathbb{C}$
$z \mapsto z^2$

In polar coordinates: $z = \rho e^{2\pi i\theta}$ $\quad \rho \in \mathbb{R}_+ = [0; +\infty)$, $\theta \in \mathbb{R}/\mathbb{Z}$.

$f(\rho e^{2\pi i \theta}) = \rho^2 e^{2\pi i (2\theta)}$ $\qquad f^n(\rho e^{2\pi i\theta}) = \rho^{2^n} \cdot e^{2\pi i (2^n \theta)}$

f acts as $\rho \mapsto \rho^2$ on the modulus, and as
$\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$
$\theta \mapsto 2\theta$ on the argument.



- if $|z_0| < 1$, then $\lim_{n\to\infty} z_n = 0$.
- all points in the unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ converge to 0

If $|z_0| > 1$, then $\lim_{n\to\infty} |z_n| = +\infty$, and z_n diverges to infinity.

(in fact, we may consider $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ the natural extension of $z \mapsto z^2$ by setting $f(\infty) = \infty$. then $\lim_{n\to\infty} z_n = \infty$).

What happens if $|z_0| = 1$?

- $z_0 = 1 \Rightarrow f(1) = 1$. (1 is called a fixed point for f.
$\mathrm{Fix}(f) = \{z \in \mathbb{C} \mid f(z) = z\}$.)
(or $\hat{\mathbb{C}}$)

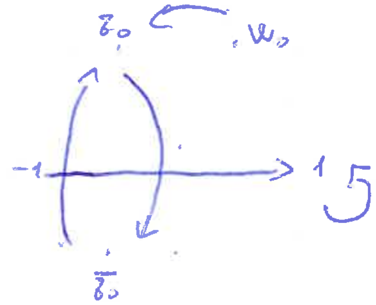
- $z_0 = -1$ $\qquad f(-1) = 1$

-1 is a prefixed point.

- $z_0 = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$. $\qquad f(z_0) = e^{\frac{4\pi i}{3}} = \bar{z}_0$, $\quad f(\bar{z}_0) = z_0$.

$e^{\frac{2\pi i}{3}}$ is a periodic point (of period 2)

$w_0 = e^{\frac{\pi i}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2} i \qquad f(w_0) = z_0$



Def: $f : X \to X$

$z_0 \in X$ is called:

- fixed point if $f(z_0) = z_0$. (set: $\mathrm{Fix}(f)$)
- periodic point if $\exists n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $f^n(z_0) = z_0$. (set $\mathrm{Per}(f)$)

  Such a n is called a period of z_0.

  The minimal of such n is called the (exact) period.
- preperiodic point: if $\exists n > m \geq 0$ s.t. $f^n(z_0) = f^m(z_0)$.

Prop: Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z^2$.

Then $z \in \partial \mathbb{D} = \{ z \in \mathbb{C} \mid |z| = 1 \}$ is preperiodic if and only if $z = e^{2\pi i \frac{p}{q}}$, $\frac{p}{q} \in \mathbb{Q}$.

Proof: The statement correspond to showing that for the map $g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $\theta \mapsto 2\theta \pmod{\mathbb{Z}}$

then θ is preperiodic $\Leftrightarrow$ $\theta \in \mathbb{Q}$.

($\Rightarrow$) θ is preperiodic $\Leftrightarrow$ $\exists n > m \geq 0$ s.t. $g^n(\theta) = g^m(\theta)$

$\Leftrightarrow \exists k \in \mathbb{Z}$ s.t. $2^n \theta = 2^m \theta + k$. $\rightsquigarrow$ $\theta = \dfrac{k}{2^n - 2^m} \in \mathbb{Q}$.

($\Leftarrow$) $\theta = \frac{p}{q} \in \frac{1}{q}\mathbb{Z}$. $\quad 2\theta = \frac{2p}{q} \in \frac{1}{q}\mathbb{Z}$

$\Rightarrow \theta_n = 2^n \cdot \theta_0 \in \frac{1}{q}\mathbb{Z}$. Quotienting by the action of $/\mathbb{Z}$, we get that $\forall n \in \mathbb{N}$, $\theta_n \in \frac{1}{q}\mathbb{Z} / \mathbb{Z}$. This set is finite; $\frac{1}{q}\mathbb{Z}/\mathbb{Z} \simeq \left\{ \frac{p}{q} \mid 0 \leq p < q,\ p \in \mathbb{N} \right\}$

This implies that $\exists n > m \geq 0$ s.t. $\theta_n = \theta_m \pmod{\mathbb{Z}}$, and θ_0 is preperiodic.

$\square$

Remark: If $\theta \in \mathbb{R} \setminus \mathbb{Q}$, ~~we~~ we always have $\# \mathcal{O}_f(e^{2i\pi\theta}) = +\infty$, but $\mathcal{O}_f(e^{2i\pi\theta})$ may be dense or not.

To see this, we can write θ in binary bases:

$\theta = 0, a_1 a_2 a_3 \dots \quad a_j \in \{0,1\} \quad 2\theta = 0, a_2 a_3 a_4 \dots$

is represented by

- Take all finite sequences of 0's and 1's, order them, then put them one after the other $\theta = 0,0100011011000\dots$

The orbit of this θ is dense.

- $\theta = 0,01001000 1 \dots \underbrace{0\dots0}_{n \text{ times}} 1 \dots$

The biggest element in the orbit is $2\theta < 0,101$, while image is not dense.

Both behaviors are dense since they are asymptotic, and we can add any finite sequence of 0 and 1 at the beginning.

To sum up:

The orbit of points in $\mathbb{D}$ varies regularly (convergence to 0).

" $\mathbb{C} \setminus \overline{\mathbb{D}}$ " " ("convergence" to ∞).

The orbit of points in $\partial\mathbb{D}$ is very chaotic: some ~~[illegible]~~ have finite orbit, some have a dense orbit in $\partial\mathbb{D}$, some don't.

Dichotomy:

Fatou set of f: $\mathcal{F}(f) \overset{\text{""}}{=} \{ z_0 \in X \mid \text{the orbit of } z \text{ varies regularly when close to } z_0 \}$

Julia set of f: $\mathcal{J}(f) = \{ z_0 \in X \mid \text{the orbit of } z \text{ varies chaotically when close to } z_0 \}$

$= X \setminus \mathcal{F}(f)$

Rmk: The proper definition of Fatou / Julia set is in terms of equicontinuity / normality of the family $\{ f^n \mid n \in \mathbb{N} \}$

Other examples:

- Tchebychev Polynomials: $T_k \in \mathbb{C}[z]$ $(\in \mathbb{Z}[z])$ satisfying

$\cos(kt) = T_k(\cos t)$ $\quad \forall t \in \mathbb{R}$ (or $\mathbb{C}$) $\quad (k \in \mathbb{N}^*)$

$T_1(z) = z$; $\quad T_2(z) = 2z^2 - 1$

Computation: $\cos(kt) = Re(e^{ikt}) = Re((e^{it})^k) = Re((\cos t + i \sin t)^k)$

$$= Re\left(\sum_{h=0}^{k} \binom{k}{h} (\cos t)^{k-h} i^h (\sin t)^h \right) = \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2l} (\cos t)^{k-2l} (-1)^l (1 - \cos^2 t)^l$$

$$\rightsquigarrow T_k(z) = \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2l} z^{k-2l} (-1)^l \cdot (1 - z^2)^l .$$

$$T_3(z) = \underline{\binom{3}{0} z^3 + \binom{3}{2} z \cdot (-1) \cdot (1 - z^2)} = 4z^3 - 3z.$$

Notice that $T = T_k$ satisfies $T^n(\cos t) = \cos(k^n t)$ $\quad \forall t \in \mathbb{C}$.

Set $I = [-1, 1]$ and $\Omega = \mathbb{C} \setminus I$

- $\forall V \subset I$ non-trivial subinterval $\exists n$ s.t. $T^n(V) = I$.
  (length$(V) > 0$)

In fact, there exists U non trivial interval in $\mathbb{R}$, $V = \cos(U)$.

$T^n(V) = T^n(\cos(U)) = \cos(k^n U) = I$ as long as

length$(k^n U) = k^n$ length $U > 2\pi$.

In particular; $I \subseteq J(T)$.

We will show that $\forall z \in \underbrace{\mathbb{C}\setminus I}_{\Omega}$, $T^n(z) \to \infty$. ($\Rightarrow J(T) = I$)

Method 1: $z \in \Omega \Leftrightarrow \exists\, w = x + iy$, $\cos w = z$ ($y \neq 0$)

$|\cos(x+iy)|^2 = |\cos x \cosh y + i \sin x \sinh y|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$

$= \cos^2 x(1+\sinh^2 y) + \sin^2 x \sinh^2 y = \cos^2 x + \sinh^2 y \geq \sinh^2 y$.

Hence $|T^n(\cos w)| = |\cos(k^n w)| \geq |\sinh(k^n y)| \to +\infty$ $\square$

Method 2: Consider $g(z) = z^k$. $\phi(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ ($g, \phi : \hat{\mathbb{C}} \circlearrowleft$)

We have: $T \circ \phi = \phi \circ g$.

In fact, if $z = e^{it}$ $t \in \mathbb{R}$ ($|z| = 1$), we have:

$$T \circ \phi(z) = T\left(\frac{e^{it} + e^{-it}}{2}\right) = T(\cos t) = \cos(kt) = \frac{e^{ikt} + e^{-ikt}}{2} = \phi(z^k).$$

By analytic continuation, $T \circ \phi = \phi \circ g$ on $\hat{\mathbb{C}}$.

We say that T and g are semiconjugated

$$\begin{tikzcd} \hat{\mathbb{C}} \arrow[r, "g"] \arrow[d, "\phi"'] & \hat{\mathbb{C}} \arrow[d, "\phi"] \\ \hat{\mathbb{C}} \arrow[r, "T"] & \hat{\mathbb{C}} \end{tikzcd}$$

Def: $f : X \to X$ and $g : Y \to Y$ are semiconjugated if there exists a holomorphic map $\Phi : Y \to X$ so that the following diagram commute;

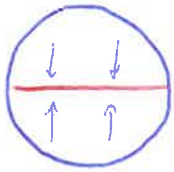
$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\Phi"'] & Y \arrow[d, "\Phi"] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

If Φ is invertible (biholomorphism), we say that f and g are conjugate, and denote $f \simeq g$.

Consider an orbit $y, g(y), g^2(y) \ldots$, and let $x = \Phi(y)$.

Then the orbit of x is given by $(\phi(g^n(y)))_{n \in \mathbb{N}}$.

In our case: $\Phi : \partial\mathbb{D} \xrightarrow{[2:1]} I$
$z \mapsto \mathrm{Re}\, z$.



$\Phi : \mathbb{C}\setminus\overline{\mathbb{D}} \xrightarrow{1:1} \Omega = \mathbb{C}\setminus I$

Moreover, $\Phi(\infty) = \infty$. It follows that $T^n(z) \to \infty$ $\forall z \in \Omega$

(since $g^n(z) \to \infty$ $\forall z \in \mathbb{C}\setminus\overline{\mathbb{D}}$).

Remk: in both cases: $J(f)$ and $F(f)$ are both <u>totally invariant</u>.
This is not a case, but general in the theory.

Def: $f : X \to X$ a (holomorphic) map. $A \subseteq X$ non-empty subset.

A is: forward invariant if: $f(A) = A$

backward invariant if $f^{-1}(A) = A$ $\Rightarrow$ if f surjective) if f injective

totally invariant if $f(A) = A = f^{-1}(A)$

• $z^2 + c$ and the Mandelbrot set.

The fact that $J(f)$ is smooth is quite special. In general, $J(f)$ has some fractal structure. Consider:

• $f(z) = z^2 - 1$.

1) If $|z| > \frac{1+\sqrt{5}}{2}$, then $f^n(z) \to \infty$

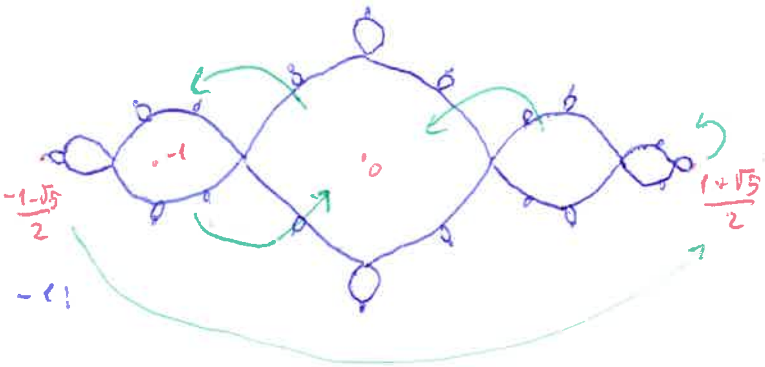
~~Assume $|z|$~~ We want to show that $\forall \lambda > 1$ $\exists \mu = \mu(\lambda) > 0$ s.t.
if $|z| \geq \mu$, then $|f(z)| \geq \lambda |z|$.

Since $\lambda > 1$, we would get $|f^n(z)| \geq \lambda^n |z| \to +\infty$ for $n \to \infty$

Now: $|f(z)| = |z^2 - 1| \geq |z|^2 - 1$

$|z|^2 - 1 \geq \lambda |z| \Leftarrow |z| \geq \frac{\lambda + \sqrt{\lambda^2 + 4}}{2} =: \mu$. We get the claim because $\mu(\lambda) \xrightarrow[\lambda \to 1]{} \frac{1+\sqrt{5}}{2}$

In this case, $F(f)$ has infinitely many connected components (all simply connected). The one containing 0, call it Ω_0, is sent to the one containing -1: Ω_{-1}, and viceversa.



$\Omega_{-1} \rightleftarrows \Omega_0 \leftarrow \Omega_1$

periodic component

preperiodic components

We will see that for $f: \hat{\mathbb{C}} \circlearrowleft$, all components are preperiodic (Sullivan)

- $f(z) = z^2 - 2$.

f is conjugated to T_2 (Tchebichev polynomial): $T_2(z) = 2z^2 - 1$

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{T_2} & \mathbb{C} \\ \Phi \downarrow & & \downarrow \Phi(z) = 2z. \\ \mathbb{C} & \xrightarrow{f} & \mathbb{C} \end{array}$$

In particular $J(f) = \Phi(J(T_2)) = \Phi([-1,1]) = [-2,2]$.

- $f(z) = z^2 - 3$.

We will show that in this case $J(f)$ is a Cantor set.

Def: $E \subset \hat{\mathbb{C}}$ is a Cantor set if it is closed, non empty, perfect (there are no isolated points) and totally disconnected (each connected component is a point)

[Classic example: $K_0 = [0,1]$, $K_1 = [0,\frac{1}{3}] \cup [\frac{2}{3}, 1]$.

$K_{n+1} = \frac{1}{3} K_n \cup \left(\frac{1}{3} K_n + \frac{2}{3}\right)$. $K_\infty = \bigcap_{n=0}^{\infty} K_n$]

Let us study how $J(f_c)$ varies when $c \in \mathbb{C}$, where $f_c(z) = z^2 + c$

It turns out that the connectivity of $J(f_c)$ depends on the behavior (the boundedness) of the critical points of f_c

Def: $f: \mathbb{C} \to \mathbb{C}$; $z \in \mathbb{C}$ is a critical point if $f'(z) = 0$. $\mathrm{Crit}(f) = \{z \mid f'(z) = 0\}$

In our case: $\mathrm{Crit}(f_z) = \{0\}$, since $f_c'(z) = 2z$.

In the examples seen, we have:

- $c=0$: $z_0 = 0 \mapsto 0 \mapsto 0 \dots$ (fixed point)
- $c=-1$: $z_0 = 0 \mapsto -1 \mapsto 0 \mapsto -1 \mapsto 0 \dots$ (periodic point)
- $c=-2$: $z_0 = 0 \mapsto -2 \mapsto 2 \mapsto 2 \mapsto 2 \dots$ (preperiodic point)
- $c=-3$: $z_0 = 0 \mapsto -3 \mapsto 6 \mapsto 33 \mapsto \dots$ (tends to ∞)

For $c=0$, we had the fixed point $z_0 = 0$, which is superattracting: $f'(0) = 0$.

Let us identify the values c for which f admits an attracting fixed point ($|f'(z_0)| < 1$).

$z^2 + c$ has two fixed points in $\mathbb{C}$ (counted with multiplicity), α and β, satisfying $z^2 - z + c = 0$. In particular $\alpha + \beta = 1$; $\alpha \cdot \beta = c$. Since $f'(z) = 2z$, this gives:

$f'(\alpha) + f'(\beta) = 2(\alpha + \beta) = 2$, and f admits at most an attracting fixed point. Say it is α. The condition is then:

$1 > |f'(\alpha)| = 2|\alpha|$. Since α is fixed, $\alpha^2 - \alpha + c = 0 \Rightarrow c = \alpha - \alpha^2$.

Hence f_c has an attracting fixed point $\Leftrightarrow c \in \{\alpha - \alpha^2 \mid |\alpha| < \frac{1}{2}\}$.

This identifies the interior of a cardioid as in the picture.

Similarly, we notice that for

$c=-1$ we had a superattracting periodic point of period 2.

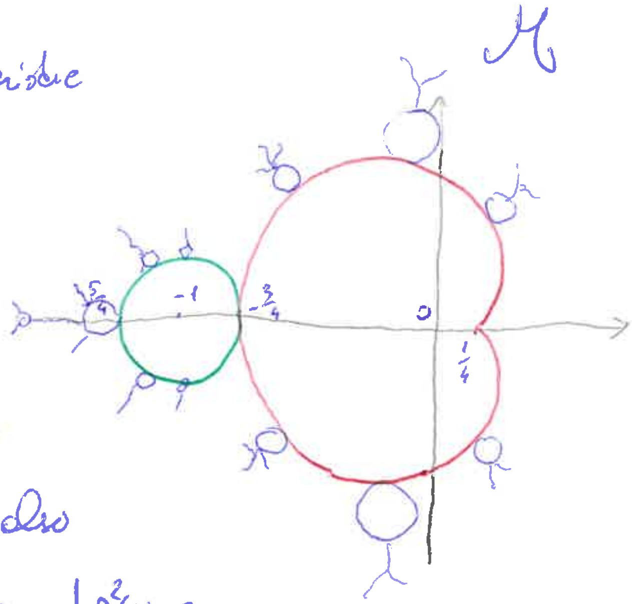
It would give a cycle

$\gamma \rightleftarrows \delta \quad \alpha\circlearrowleft \quad \beta\circlearrowleft$

It would satisfy $P^{\circ 2}(z)-z=0$.

Since α and β are fixed, they also satisfy $P^2(z)-z=0$, and $P(z)-z \mid P^2(z)-z$.

$$P^2(z)-z=(P(z)-z)\underbrace{(z^2+z+1+c)}_{(z-\gamma)(z-\delta)}$$



We want to impose $|(P^2)'(\gamma)|<1$ (and $|(P^c)'(\delta)|<1$)

$$(P^2)'(\gamma)=P'(\gamma)\cdot P'(\delta)=4\gamma\delta=4(1+c).$$

Hence the condition is $\{c \mid |1+c|<\frac{1}{4}\}$ which is a disc $D(-1;\frac{1}{4})$.

In both cases, the critical point $z_0=0$ is attracted by the attracting fixed/periodic points in the sense that $f_c^n(0)\to\alpha$, $f_c^{2n}(0)\to\gamma$ or δ.

It is natural to consider the set:

$$\mathcal{M}=\{c\in\mathbb{C} \mid \underbrace{f_c^n(0) \text{ is bounded}}_{(\not\to\infty)}\}.$$

$\leadsto$ Bifurcation theory.

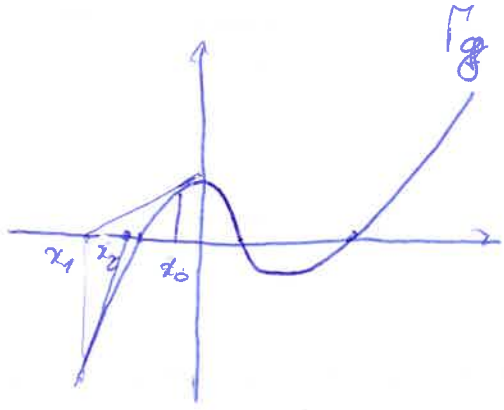
We will see that $J(f_c)$ is connected $\Leftrightarrow c\in\mathcal{M}$.

Newton's approximation method

It is a method to find zeroes of functions $g : \mathbb{R} \to \mathbb{R}$

If g is differentiable;

Start from x_0 a guess for a solution of $g = 0$.



Geometrically, given a point x_n constructed by the algorithm, consider $(x_n, g(x_n)) = p_n \in \Gamma_g$ and the tangent of Γ_g at p_n, of equation $y - g(x_n) = g'(x_n)(x - x_n)$

Consider the intersection with $y = 0$, and set $x_{n+1} = x_n - \dfrac{g(x_n)}{g'(x_n)}$

Hence $x_{n+1} = f(x_n)$, where $f(x) = x - \dfrac{g(x)}{g'(x)}$.

Notice that $f(x) = x \iff g(x) = 0$

One can show that if $\tilde{x}$ is a zero of g and x_0 is close enough to g, then $x_n = f^n(x_0) \underset{n\to\infty}{\longrightarrow} \tilde{x}$. (exponentially fast).

Consider this algorithm for holomorphic maps (actually, polynomials):

$g : \mathbb{C} \to \mathbb{C}$ $\quad g(z) = P(z)$, $\quad P \in \mathbb{C}[z]$.

$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$; $\quad f(z) = z - \dfrac{P(z)}{P'(z)}$.

We want to show that $\forall \tilde{z} \in \mathbb{C}$, $P(\tilde{z}) = 0$, is an attracting fixed point for f.

Let $\tilde{z}$ be a zero of order $m \geq 1$. Then $P(z) = (z - \tilde{z})^m h(z)$, $h(\tilde{z}) \neq 0$.

$P'(z) = m(z - \tilde{z}_0)^{m-1} h(z) + (z - \tilde{z})^m h'(z) = m(z - \tilde{z})^{m-1}(h_1(z))$ $\quad h_1(\tilde{z}) = h(\tilde{z}) \neq 0$.

$P''(z) = m(m-1)(z-\tilde{z})^{m-2}\cdot h_2(z)$, $\quad h_2(\tilde{z}) = h_1(\tilde{z}) = h_0(\tilde{z}) \neq 0$

$$\leadsto f'(z) = 1 - \frac{(P'(z))^2 - P(z)P''(z)}{(P'(z))^2} = \frac{P(z)\cdot P''(z)}{(P'(z))^2} = \frac{(z-\tilde{z})^{2m-2}\cdot h(z)\cdot h_2(z)\cdot m(m-1)}{(z-\tilde{z})^{2m-2}(h_1(z))^2\cdot m^2}$$

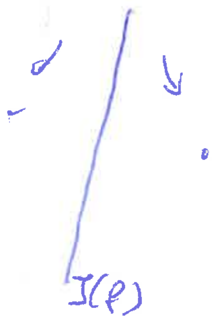
and $f'(\tilde{z}) = \frac{m-1}{m} = 1 - \frac{1}{m}$.

Hence $\tilde{z}$ is an attracting fixed point for f (superattracting if $m=1$)

and $\exists\, \varepsilon > 0$ s.t. $f^n(z) \to \tilde{z}$ $\forall z,\ |z-\tilde{z}| < \varepsilon$.

Problem: finding a good starting point for the algorithm to start.

if: $\deg P = 2$ one can show that the algorithm converges to the solution that is the closest:



For higher degrees, $J(f)$ is much more complicated.

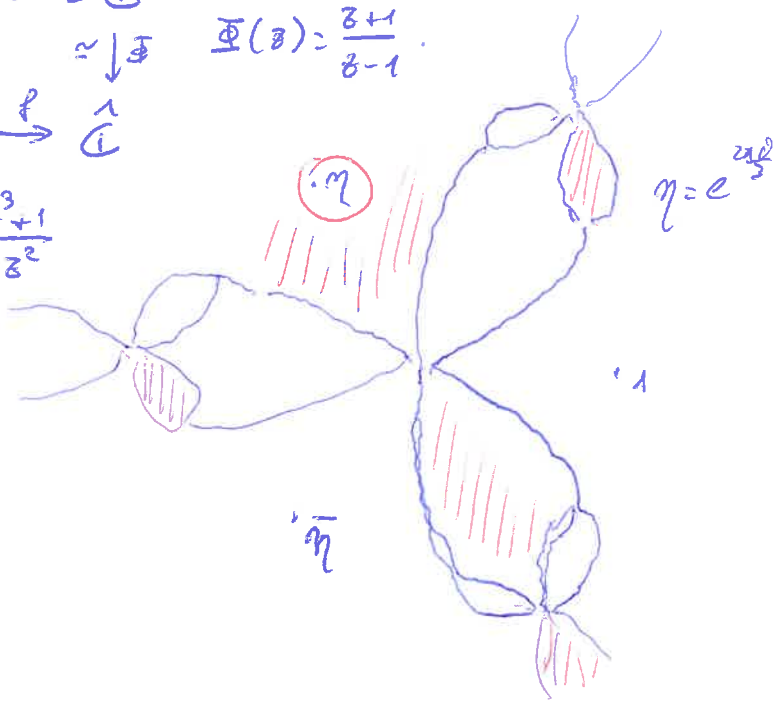
Ex: $P(z) = z^2 - 1$

$\leadsto f(z) = z - \frac{z^2-1}{2z} = \frac{z^2+1}{2z}$.

$f(z)$ is conjugated to $z \mapsto z^2$

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{z\mapsto z^2} & \hat{\mathbb{C}} \\ \Phi \downarrow \cong & & \cong \downarrow \Phi \\ \hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}} \end{array} \qquad \Phi(z) = \frac{z+1}{z-1}.$$

Ex: $P(z) = z^3 - 1$; $f(z) = z - \frac{z^3-1}{3z^2} = \frac{2z^3+1}{3z^2}$



· Questions answered recently:

given P, find $z_1 \dots z_N$ ($N \gg \deg P$)

so that $f^n z_j \to$ solutions of $\{P=0\}$.